

AN OPERATOR PRODUCT INEQUALITIES FOR POLYNOMIALS

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ABSTRACT. Let $P(z)$ be a polynomial of degree $n \geq 1$. In this paper we define an operator B , as following,

$$B[P(z)] := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!},$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + c(n, 1)\lambda_1 z + c(n, 2)\lambda_2 z^2$$

lie in half plane

$$|z| \leq \left| z - \frac{n}{2} \right|$$

and obtain a new generalization of some well-known results.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let P_n be the class of polynomials of degree at most n then,

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

and

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|, \quad R > 1. \quad (1.2)$$

Inequality (1.1) is an immediate consequence of Bernstein's theorem on the derivative of a polynomial (see[5]). Inequality (1.2) is a simple deduction from the maximum modulus principle (see[11]). If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then the inequality (1.1) and (1.2) can be respectively replaced by following [9,1]

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \quad (1.3)$$

and

$$\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \quad R > 1. \quad (1.4)$$

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Recently Aziz and Rather [3] have investigated the dependence of

$$\left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \text{ on } \max_{|z|=1} |P(z)|$$

for all real and complex α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$, in fact, they proved

Theorem A. If $P(z)$ is a polynomial of degree n , then for all real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,

$$\begin{aligned} & \left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \\ & \leq \left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \max_{|z|=1} |P(z)| \text{ For } |z| \geq 1 \end{aligned} \quad (1.5)$$

and

$$\begin{aligned} & \left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \\ & + \left| Q(Rz) - \alpha Q(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} Q(z) \right| \\ & \leq \left[\left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \right. \\ & \left. + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| \right] \max_{|z|=1} |P(z)| \end{aligned} \quad (1.6)$$

for $|z| \geq 1$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$. The results are sharp and equality in (1.5) and (1.6) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$. For the class of polynomial having no zeros in $|z| < 1$, we have the following result due to Aziz and Rather which is a generalization of inequality (1.4).

Theorem B. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for all real and complex number α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,

$$\begin{aligned} & \left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \\ & \leq \frac{1}{2} \left[\left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \right. \\ & \left. + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| \right] \max_{|z|=1} |P(z)| \end{aligned} \quad (1.7)$$

for $|z| > 1$. Equality in (1.7) occurs for $P(z) = z^n + 1$.

In this paper, we consider on operator B which carries $P \in P_n$ in to

$$B[P(z)] := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \quad (1.8)$$

where λ_0 , λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + c(n, 1)\lambda_1 z + c(n, 2)\lambda_2 z^2 \quad (1.9)$$

lie in the half plane

$$|z| \leq \left| z - \frac{n}{2} \right| \quad (1.10)$$

and prove the following generalization of theorem A and B thus as well of inequalities (1.1) and (1.2).

Theorem 1. If $P(z)$ is a polynomial of degree n , then for real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & \leq \left| (R^n - \alpha) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |B[z^n]| \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1. \end{aligned} \quad (1.11)$$

Equality holds in (1.11) for $P(z) = \lambda z^n, \lambda \neq 0$.

Remark 1. For $\lambda_0 = \lambda_2 = 0$ in (1.11) and note that in this case all the zeros of $u(z)$ defined by (1.9) lie in (1.10), we get

$$\begin{aligned} & \left| RP'(Rz) - \alpha P'(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P'(z) \right| \\ & \leq n \left| (R^n - \alpha) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^{n-1} \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1. \end{aligned} \quad (1.12)$$

Equality holds in (1.12) for $P(z) = \lambda z^n, \lambda \neq 0$. If we take $\beta = 0$, $\alpha = 1$ and dividing the two sides of (1.12) by $R - 1$ and make $R \rightarrow 1$, we get

$$\max_{|z|=r \geq 1} |zP''(z) + P'(z)| \leq n^2 r^{n-1} \max_{|z|=1} |P(z)|. \quad (1.13)$$

Equality holds in (1.13) for $P(z) = \lambda z^n, \lambda \neq 0$. for $\alpha = \beta = 0$ and $R = 1$, inequality (1.12) gives

$$|P'(z)| \leq n|z|^{n-1} \max_{|z|=1} |P(z)|. \text{ for } |z| \geq 1. \quad (1.14)$$

which in particular gives inequality (1.1).

Remark 2. For $\lambda_1 = \lambda_2 = 0$ in theorem 1 reduces to inequality (1.5). Next as a application of theorem 1, we prove the following theorem which is a generalization of a results prove by Rahman [10] , Jain [6], Aziz and Rather [3].

Theorem 2. If $P(z)$ is a polynomial of degree n , then for real or complex number α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$,

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & + \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right| \\ & \leq \left[R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right] |B[z^n]| \\ & + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |\lambda_0| \max_{|z|=1} |P(z)| \end{aligned} \quad (1.15)$$

for $|z| \geq 1$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$. If we take $\lambda_0 = \lambda_2 = \beta = 0$ and $\alpha = 1$ in (1.15), we obtain the following result.

Corollary 1. If $P(z)$ is a polynomial of degree n , then for all real or complex number α with $|\alpha| \leq 1$ and $R \geq 1$,

$$\begin{aligned} & |RP'(Rz) - P'(z)| + |RQ'(Rz) - Q'(z)| \\ & \leq n(R^n - 1)|z|^{n-1} \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \end{aligned} \quad (1.16)$$

Equality holds in (1.16) for $P(z) = \lambda z^n$, $\lambda \neq 0$. Theorem 2 includes a result due to Rahman [10] as a special case for $\lambda_1 = \lambda_2 = \alpha = \beta = 0$ where as inequality (1.15) reduces to a result due to Jain [6, Theorem 1] for $\lambda_1 = \lambda_2 = \alpha = 0$. For $\lambda_1 = \lambda_2 = 0$, inequality (1.15) reduces to inequality (1.6).

Lastly, for class of polynomial having no zeros in $|z| < 1$, we prove the following generalization of theorem B.

Theorem 3. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for all real and complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$, then

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & \leq \frac{1}{2} \left[R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right] |B[z^n]| \\ & + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |\lambda_0| \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \end{aligned} \quad (1.17)$$

Equality holds in (1.17) for $P(z) = z^n + 1$. If we take $\alpha = \beta = 0$ in theorem 3, we get the following result.

Corollary 2. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for $R \geq 1$,

$$|B[P(Rz)]| \leq \frac{1}{2} \{R^n |B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \quad (1.18)$$

The result is sharp and equality holds for $P(z) = z^n + 1$. For $R = 1$, inequality (1.18) reduces to a results due to Shah and Liman [12].

Remark 3. Theorem 3 includes some well-known inequalities as special case. For example inequality (1.17) reduces to a result due to Aziz and Rather [4] for $\lambda_1 = \lambda_2 = \beta = 0$. For $\lambda_1 = \lambda_2 = \alpha = 0$ inequality (1.17) reduces to result due to Jain [7] where as for $\lambda_1 = \lambda_2 = 0$ inequality (1.18) reduces to

$$|P(Rz)| \leq \frac{1}{2} \{R^n + 1\} \max_{|z|=1} |P(z)| \quad R \geq 1.$$

If we take $\lambda_0 = \lambda_2 = \alpha = \beta = 0$, inequality (1.17) reduces to inequality (1.3).

2. LEMMAS

For the proofs of the theorems, we need the following lemmas.

Lemma 1. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ ($k \leq 1$), then for every $R > 1$,

$$|P(Rz)| \geq \left(\frac{R+k}{1+k} \right)^n |P(z)| \quad \text{for } |z| = 1. \quad (2.1)$$

This lemma was proved by Aziz [2]. If we take $k = 1$ and $P(z) \neq 0$ in $|z| \geq 1$ in the above lemma, then we can prove the following lemma.

Lemma 2. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| < 1$, then for every $R > 1$

$$|P(Rz)| > \left(\frac{R+1}{2} \right)^n |P(z)| \quad \text{for } |z| = 1. \quad (2.2)$$

Proof . Let for $|z_i| > 1$ ($i = 1, 2, \dots, n$), $F(z) = z^n \overline{P(\frac{1}{\bar{z}})} = c \prod_{i=1}^n (z - z_i)$ for $z = \rho e^{i\theta}$ ($0 \leq \rho < 1$), we have $|\frac{z - z_i}{e^{i\theta} - \bar{z}_i}| > \frac{\rho+1}{2}$ ($-\pi \leq \theta \leq \pi$), therefor

$$\left| \frac{F(\rho z)}{F(z)} \right| > \left(\frac{\rho+1}{2} \right)^n, \quad |z| = 1$$

finally for $\rho = \frac{1}{R}$, we get the desired result.

Lemma 3. If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[P(z)]$ also lie in the circle $|z| \leq 1$.

Lemma 4. If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$, in $|z| < 1$, then

$$|B[P(z)]| \leq |B[Q(z)]| \quad \text{for} \quad |z| \geq 1. \quad (2.3)$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Lemma 5. If $P(z)$ is a polynomial of degree n , then for $|z| \geq 1$,

$$|B[P(z)]| + |B[Q(z)]| \leq \{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)|. \quad (2.4)$$

The above three lemmas are due to Shah and Liman [12].

Lemma 6. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for all real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$,

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & \leq \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right| \end{aligned} \quad (2.5)$$

for $|z| \geq 1$ where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Proof. If $P(z) \neq 0$ in $|z| < 1$, then by lemma 4 we have $|B[P(z)]| \leq |B[Q(z)]|$ for $|z| \geq 1$ and hence for $R = 1$, we have nothing to prove. For $R > 1$, since $|P(z)| = |Q(z)|$ for $|z| = 1$, it follows by Rouché's theorem that for every real or complex number λ with $|\lambda| > 1$, the polynomial $T(z) = P(z) - \lambda Q(z)$ does not vanish in $|z| > 1$, with at least one zero in $|z| < 1$. Let $T(z) = (z - re^{i\delta})F(z)$ where $r < 1$ and $F(z)$ is a polynomial of degree $n - 1$ having no zeros in $|z| > 1$. Applying lemma 1 with $k = 1$, for every $R > 1, 0 \leq \theta \leq 2\pi$

$$\begin{aligned} |T(Re^{i\theta})| & \geq |Re^{i\theta} - re^{i\delta}| \left(\frac{R+1}{2} \right)^{n-1} |F(e^{i\theta})| \\ & = \left(\frac{R+1}{2} \right)^{n-1} \left| \frac{Re^{i\theta} - re^{i\delta}}{e^{i\theta} - re^{i\delta}} \right| |(e^{i\theta} - re^{i\delta})F(e^{i\theta})| \\ & \geq \left(\frac{R+1}{2} \right)^{n-1} \left(\frac{R+r}{1+r} \right) |T(e^{i\theta})| \end{aligned}$$

or

$$\left(\frac{r+1}{R+1} \right) |T(Re^{i\theta})| \geq \left(\frac{R+1}{2} \right)^{n-1} |T(e^{i\theta})|, R > 1 \quad \text{and} \quad 0 \leq \theta \leq 2\pi, \quad (2.6)$$

since $R > 1 > r$, hence $T(Re^{i\theta}) \neq 0$ and $(\frac{2}{R+1}) > (\frac{r+1}{R+r})$, from inequality (2.6), we have

$$|T(Rz)| > \left(\frac{R+1}{2}\right)^n |T(z)|, \quad |z| = 1, \quad R > 1. \quad (2.7)$$

Hence for every real or complex number with $|\alpha| \leq 1$, we have

$$\begin{aligned} |T(Rz) - \alpha T(z)| &\geq |T(Rz)| - |\alpha| |T(z)| \\ &> \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} |T(z)| \quad \text{for } |z| = 1 \text{ and } R > 1. \end{aligned} \quad (2.8)$$

Since $T(Re^{i\theta}) \neq 0$ and $(\frac{R+1}{2})^n > 1$, hance from inequality (2.7), we have

$$|T(Re^{i\theta})| > (|T(e^{i\theta})|), \quad \text{for } R > 1 \text{ and } 0 \leq \theta \leq 2\pi$$

or

$$|T(Rz)| > (|T(z)|), \quad \text{for } |z| = 1 \text{ and } R > 1.$$

Since all the zeros of $T(Rz)$ lie in $|z| < 1$, it follows (by Rouché's theorem for $|\alpha| \leq 1$) that the polynomial $T(Rz) - \alpha T(z)$ does not vanish in $|z| \geq 1$. Hence from inequality (2.7)(by Rouché theorem for $|\beta| \leq 1$), we have the polynomial

$$S(z) = T(Rz) - \alpha T(z) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} T(z)$$

has all its zeros in $|z| < 1$. Therefore, by lemma 3, all the zeros of $B[S(z)]$ lie in $|z| < 1$. Replacing $T(z)$ by $P(z) - \lambda Q(z)$ and since B is liner, it follows that

$$\begin{aligned} &\left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[P(z)] \right| \\ &\leq \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[Q(z)] \right| \end{aligned} \quad (2.9)$$

for $|z| \geq 1$. If this is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\begin{aligned} &\left| B[P(Rz_0)] - \alpha B[P(z_0)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[P(z_0)] \right| \\ &> \left| B[Q(Rz_0)] - \alpha B[Q(z_0)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[Q(z_0)] \right| \end{aligned}$$

since all the zeros of $Q(z)$ lie in $|z| \leq 1$, hence (some as $T(z)$) all the zeros of

$$B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[Q(z)] \quad \text{lie in } |z| < 1$$

for all real or complex α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$. Therefore

$$B[Q(Rz_0)] - \alpha B[Q(z_0)] + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} B[Q(z_0)] \neq 0 \quad \text{whit } |z_0| \geq 1 \text{ we take}$$

$$\lambda = \frac{B[P(Rz_0)] - \alpha B[P(z_0)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z_0)]}{B[Q(Rz_0)] - \alpha B[Q(z_0)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z_0)]}$$

so that $|\lambda| > 1$ and for this value λ , $B[S(z_0)] = 0$ for $|z_0| \geq 1$, which contradicts the fact that all the zeros of $B[S(z)]$ lie in $|z| < 1$. This proves the desired result.

3. PROOF OF THEOREMS

Proof of Theorem 1. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M|z^n|$ for $|z| = 1$.

If λ is any real or complex number with $|\lambda| > 1$, then by Rouché's theorem the polynomial $P_1(z) = P(z) - \lambda M z^n$ has all its zeros in $|z| < 1$, and by lemma 3, all the zeros of $B[P_1(z)]$ lie in $|z| < 1$, it follows that

$$|B[P(z)]| \leq M|B[z^n]| \quad \text{for } |z| \geq 1$$

If this is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$|B[P(z)]| > M|B[z^n]|$$

since $B[z^n] \neq 0$ for $|z| \geq 1$, we take $\lambda = \frac{B[P(z_0)]}{MB[z_0^n]}$ so that $|\lambda| > 1$ and for this value of λ , $B[P_1(z_0)] = 0$ for $|z_0| \geq 1$, which contradicts the fact that all the zeros of $B[P_1(z)]$ lie in $|z| < 1$. This completes the proof for $R = 1$. Now we assume $R > 1$. On applying lemma 2, we get

$$|P_1(Rz)| > \left(\frac{R+1}{2} \right)^n |P_1(z)|, \quad |z| = 1 \text{ and } R > 1. \quad (3.1)$$

Since $P_1(Re^{i\theta}) \neq 0$, $0 \leq \theta \leq 2\pi$ and $\left(\frac{R+1}{2} \right)^n > 1$ from above inequality we have $|P_1(Re^{i\theta})| > |P_1(e^{i\theta})|$, $R > 1$.

Equivalently,

$$|P_1(Rz)| > |P(z)|, \quad \text{for } |z| = 1 \text{ and } R > 1.$$

For every real and complex number α with $|\alpha| \leq 1$ and using inequality (3.1), we have

$$\begin{aligned} |P_1(Rz) - \alpha P_1(z)| &\geq |P_1(Rz)| - |\alpha| |P_1(z)| \\ &> \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} |P_1(z)| \quad \text{for } |z| = 1 \text{ and } R > 1. \end{aligned}$$

Now using the arguments similar to those used in lemma 6, the theorem follows.

Proof of Theorem 2. The result is trivial if $(R = 1)$ (lemma 5), so we suppose that $R > 1$. If $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. Now for λ with $|\lambda| > 1$, we have the polynomial $W(z) = P(z) + \lambda M$ has no zeros in $|z| < 1$ and on applying lemma 6, we get for $|z| \geq 1$ and $R > 1$,

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right. \\ & \quad \left. + \lambda \left[1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right] \lambda_0 M \right| \\ & \leq \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right. \\ & \quad \left. + \bar{\lambda} \left[R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right] B[z^n] M \right| \end{aligned} \quad (3.2)$$

where $|\alpha| \leq 1$, $|\beta| \leq 1$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$. choosing the argument of λ , which is possible by (1.11) such that

$$\begin{aligned} & \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right. \\ & \quad \left. + \bar{\lambda} \left[R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right] M B[z^n] \right| \\ & = |\lambda| \left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| M |B[z^n]| \\ & \quad - \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right| \end{aligned}$$

we get from (3.2)

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & + |\lambda| |\lambda_0| \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| M \\ & \leq |\lambda| \left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| M |B[z^n]| \\ & - \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right| \end{aligned} \quad (3.3)$$

for $|z| \geq 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$, and $R > 1$, making $|\lambda| \rightarrow 1$ in (3.3), we get (1.15). This completes the proof of theorem 2.

Proof of Theorem 3. By hypothesis $P(z)$ does not vanish in $|z| < 1$, therefore by lemma 6 we have

$$\begin{aligned} & \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & \leq \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right| \end{aligned}$$

or

$$\begin{aligned} & 2 \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & \leq \left| B[P(Rz)] - \alpha B[P(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[P(z)] \right| \\ & + \left| B[Q(Rz)] - \alpha B[Q(z)] + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} B[Q(z)] \right|. \end{aligned} \quad (3.4)$$

On combining (3.4) with (1.15), the proof of theorem 3 is complete.

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